ANOVA (Analysis of variance)

QUALITY TOOLS
Anova

Description of ANOVAs: Analysis of Variance (ANOVA) is a **generalized statistical technique** used to analyze sample variances to obtain information on comparing multiple population means. This technique is consisted of several fundamental statistical concepts (hypothesis testing, F-test). The confusing part is that we are analyzing variances to learn about the means (averages). In its simplest form such as One-Factor ANOVA (1F-ANOVA) or Two Factor (2F-ANOVA), these designs can be considered as a simple Design of Experiments (DOE) as one is examining the effects of one to two factors on a responsible variable.

There are several terms/jargons that need to be defined to learn about ANOVA.

Factor – a quantitative or a qualitative variable that is expected to exert an impact/effect on a quality characteristic.

Response Variable (RV) – the quality characteristic under investigation.

Sum-of-Squares (SS) – the sum of squared deviations

When to use the ANOVAs: ANOVA is commonly used to study the effect of one or two factors on a response variable. However in practice, it is more useful to study a lot of factors on one or more response variables together and DOE is used in those complex cases. But the concept of ANOVA is employed in DOE technique for data analysis.

How to use the ANOVAs: It is easier to use examples to illustrate the data analysis used in ANOVA. The reader must be familiar with hypothesis testing and the F-test in order to understand the concept of ANOVA.
In ANOVA, we need to understand that the variation, as observed from the experimental data, comes from two main sources – Factors under study (called Tested Factor or Factor) and from Experimental Error (error). A factor is considered statistically significant (or significant) if its effect on the RV is much larger as compared to the effect arising from error.

Under the **Null Hypothesis** (H_0), the observed variation from all of the experimental data recorded on RV is due to experimental error only. This means that even the factor under study exerts no effect on the RV. The Alternate Hypothesis (H_1) stipulates that the factor exerts a significant effect on the RV.
Any conclusion is derived through the analysis by the F-test. The F-test is set up in such a way that the variation from the factor is divided by the variation from error. Therefore, if the calculated F ratio (F_{cal}) is larger than the tabulated F value (F_{tab}), the factor under

study is considered to be significant (reject the Null Hypothesis). If the calculated F ratio (F_{cal}) is smaller than the tabulated F value (F_{tab}), the factor under study is considered to be NOT significant (accept the Null Hypothesis)

Tips on use of ANOVAs: When the number of factors increase, the analysis and calculations will be more complex. For example, in a 2-Factor ANOVA, the interaction of the 2-Factors also becomes a Factor. Therefore, there will be 3 separate hypothesis that needs to be tested. In the multi-factor case, the classical Factorial Designs approach will be employed.

Application of ANOVAs: **Case 1: 1F-ANOVA**

We assume that there are three types of plastics used to make plastic cups. The quality characteristic under study will be the inside cup opening or diameter (RV). The Factor in this example will be Plastic (a qualitative variable). If we wish to study a quantitative variable, we can study temperature, pressure, etc. on their effects on the RV. Since we have 3 “types” of plastics, we use the term “**level**” to describe each type of plastic. Our interest is to find out if the cup diameter (in cm) remains the same by using the three types of plastic (H_0 is correct) or if the plastics will result in different diameter (H_1 is correct).

In the calculation, we obtain the Total Sum-of-Square (SST), which is considered to be the “total variation” (in a descriptive manner) of all of the experimental data. This SST will be “partitioned” into 2 main components –one component as attributed to the factor and the other component attributed to error. Then we will calculate the total degrees of freedom (υ_T) and we partition this term into 2 components in a way similar to the SS situation.

Experimental Data for Plastic Study

Factor = Plastic, PS = polystyrene, PP = polypropylene, PE = polyethylene, RV = cup diameter in inches					
Level	Replicate 1	Replicate 2	Replicate 3	Total	Average
1 = PS	2.0	2.2	1.8	6.0	2.0
2 = PP	2.2	2.0	2.4	6.6	2.2
3 = PE	2.9	2.8	2.7	8.4	2.8
			Grand Total	21.0	

ANOVA Calculations Using Sum-of-Square Method

We use T to specify a row or column total.

$T_i.$ = total of the i^{th} row, the dot indicating that the total includes all column values in that row (e.g. $T_2.$ = total of all values in the second row)

$T..$ = total of all observations

X_{ij} = Value of each individual datum in the Experimental Result

ΣX_{ij}^2 = Square each individual datum first and then sum over all of these squared values

n = total number of measurements

c = number of columns

r = number of rows

Without much derivation, we arrive at the following formulae to obtain SST (total sum-of-square)

$$SST = \Sigma X_{ij}^2 - \frac{(T..)^2}{n}$$

Applying the above equation to our data in the Plastic Study, we obtain

$$SST = (2^2 + 2.2^2 + 1.8^2 + ... + 2.9^2 + 2.8^2 + 2.7^2) - (21)^2 /9$$
$$= 50.22 - 49.00 = 1.22$$

As we said earlier, this SST is to be partitioned into 2 components – one for the Factor and one for error. We can define a Sum of Square for Factor (SSF) as follows;
(Note that the Factor Effect is also known as the Row Effect in some books since the different levels of the factor is arranged in rows. The symbol SSR is used instead in those cases)

$$SSF = \frac{(T_i .)^2}{c} - \frac{(T..)^2}{n}$$

$$SSF = (6^2 + 6.6^2 + 8.4^2) / 3 - 21^2 /9 = 50.04 - 49.00 = 1.04$$

To find the Sum of Square Term for Error (SSE) , we have to go back to the original data and calculate the deviations of all the data from the corresponding row average. Thus

$$SSE = \Sigma X_{ij}^2 - \frac{T_i.^2}{c}$$

$$SSE = (2^2 + 2.2^2 + 1.8^2 + ... + 2.9^2 + 2.8^2 + 2.7^2) - (6^2 + 6.6^2 + 8.4^2) / 3$$
$$= 50.22 - 50.04 = 0.18$$

(We could have used the additive properties of Sum-of-Square by obtaining SSE simply subtracting SSF from SST. SSE can be obtained by 1.22 – 1.04 = 0.18)

The next step is to calculate the degree of freedom term for each Sum of Square Term.

ν_T for SST = n - 1 = 9 - 1 = 8

ν_R for SSR = r - 1 = 3 -1 = 2

ν_E for SSE = r(c - 1) = 3(3-1) = 6

We then calculate F statistic as follows;

$$F_{cal} = \frac{\text{Mean Square Factor (MSF)}}{\text{Mean Square of Error (MSE)}}$$

= 0.52/0.03 = 17.33

$F_{tab} = F_{v1, v2, \alpha} = F_{2, 6, 0.05} = 5.1433$

The next step is to divide each Sum of Square Term by its corresponding degree of freedom term to obtain the Mean Square (MS) Term for each Source of Variation. We then assemble all of the Terms into a "SUMMARY ANOVA TABLE"

ANOVA Summary Table

Sources of Variation	SS	dF	MS	F_{cal}	F_{tab}
Factor (plastic)	1.04	2	0.52	17.33	5.1433
Error	0.18	6	0.03		
Total	1.22	8			

Conclusion:

Since F_{cal} is > F_{tab}, , we will accept the alternate hypothesis and conclude that Plastic is a significant factor. It means each kind of plastic will behave differently. If a smaller cup diameter is preferred, then we will prefer PS as the average diameter is the smallest.
(If the null hypothesis turns out to be true, it means all the 3 different plastics behave similarly as far as the cup diameter is concerned. This means that the user can consider other practical issues such as price of the plastics, viscosity, ease of use, etc. and make the final decision to decide on which plastic is best.)

Reference: Douglas Montgomery, "Design and Analysis of Experiments", John Wiley & Sons, 1991